CHAPTER 2

THE TIDE-RAISING FORCES

2.1 Introduction

It was explained in Chapter I that the local tide results from the superposition of long waves of tidal frequencies generated throughout the ocean by the tide-raising forces of the moon and the sun. It remains to investigate these forces, particularly with a view to determining the frequencies that characterize their fluctuations. It has been reasonably assumed, and later established by experience, that these are also the frequencies of the tide waves generated in the ocean, and so are the main frequencies present in the fluctuations of the local tide. Shallow water distortion, however, may be expected to add multiples and combinations of these frequencies (over-tides) to the spectrum of a coastal tide. Fluctuation in the tide or in the tidal force at a particular frequency is called the *harmonic constituent* at that frequency. The amplitudes and phaselags of the constituents are the *harmonic constants* of the tide, the phaselag usually being referred to the phase of the corresponding constituent in the tide-raising force at Greenwich. While it may be expected that the harmonic constituents present in the spectrum of the tide-raising forces will be present in the spectrum of the local tide, it should not be expected that they will be present in the same proportion or with the same phase relation. This is because ocean basins and coastal embayments are more nearly resonant at some tidal frequencies than at others, because nodes and amphidromes occur at different locations for constituents of different frequencies, and because processes such as the transfer of energy from surface tide to internal tide may be frequency selective in different situations.

The tide-raising forces are simply the portions of the moon's and the sun's gravitational attraction that are unbalanced by the centripetal (centrally directed) acceleration of the earth in its orbital motions. At the centre of mass of the earth, and only at this point, there is an exact balance between the gravitational attractions and the centripetal accelerations, this being the condition for orbital motion. Earth gravity, which includes the centrifugal force due to rotation of the earth on its axis, determines the shape of level surfaces and hence the shape of the mean level of the sea; but it does not contribute to the tide-raising forces because it does not vary with time. Although, as we shall see later, the moon has more effect on the tide than does the sun, it will be convenient to consider the sun's contribution first, since the orbital parameters are easier to envisage for the earth-sun system.

2.2. Sun's tide-raising force

In this section we require Newton's laws of motion and of universal gravitation, and an understanding of centripetal acceleration. The law of motion states that the acceleration of a body equals the force acting on it per unit mass, or

$$(2.2.1)\quad \text{acceleration} = \frac{\text{force}}{\text{mass}}$$

The law of universal gravitation states that a body of mass M exerts a gravitational attraction on a unit mass at a distance r of

$$(2.2.2)\quad F_g = \frac{GM}{r^2}$$

in which G is the universal gravitational constant. The centripetal acceleration is the acceleration of a body toward the centre of curvature of the path along which it is moving, and for a body with velocity v along a path with radius of curvature r, it is

$$A_c = \frac{v^2}{r} .$$

Let us now compare the gravitational attraction of the sun on the earth to that of the moon on the earth. The mass of the sun is 27 million times that of the moon, and the distance of the sun from the earth is 390 times that of the moon. Using this information in equation 2.2.2 gives

$$\frac{F_g(\text{sun})}{F_g(\text{moon})} = \frac{27 \times 10^6}{390^2} = 178$$

so the gravitational attraction of the sun on the earth is 178 times that of the moon. This may at first seem surprising since we know the moon to be more effective in producing tides; but it is only the portion of the gravitational force not balanced by the centripetal acceleration in the earth's orbital motion that produces tides. This unbalanced portion will shortly be shown to be proportional to the inverse cube rather than the inverse square of the distance from the earth, but still proportional to the mass as in equation 2.2.2. Thus, the tide-raising forces of the sun are about 178/390 = 0.46 times those of the moon.

Figure 13 depicts a portion of the earth's orbit around the sun, with the cross section through the earth greatly exaggerated with respect to the sun's size and distance. Since the acceleration related to the earth's axial rotation is already accounted for in earth gravity, the earth should be thought of here as maintaining a fixed orientation in space during its revolution about the sun; thus, each part of the earth experiences the same centripetal acceleration toward the sun. In particular, the centripetal acceleration at the centre of the earth, *0*, is exactly equal to the sun's gravitational attraction at that point, this being the condition for orbital motion. The centripetal acceleration, being everywhere constant, is therefore everywhere equal to the gravitational attraction at the centre, GS/r^2, where S is the sun's mass and r its distance from the centre of the earth. At a point such as A, that is closer to the sun, the gravitational attraction is greater than at the centre, *0*, and so has an unbalanced component that attempts to accelerate a mass at A, away from *0* and toward the sun. At a point such as A', that is farther from the sun, the gravitational attraction is less than at *0*, and the unbalanced component attempts to accelerate a mass at A' away from *0* and away from the sun. At B and B' in Fig. 13, the gravitational attraction has almost the same magnitude as at *0*, but is directed toward the sun along a slightly different line, so that the unbalanced components are both acting toward *0*. These unbalanced components of gravitational attraction are the sun's tide-raising forces. At A, A', B, and B' they are vertical, but at intermediate points they are inclined to the vertical. At four of the intermediate points the forces are entirely horizontal. The horizontal components of the tide-raising forces are called the

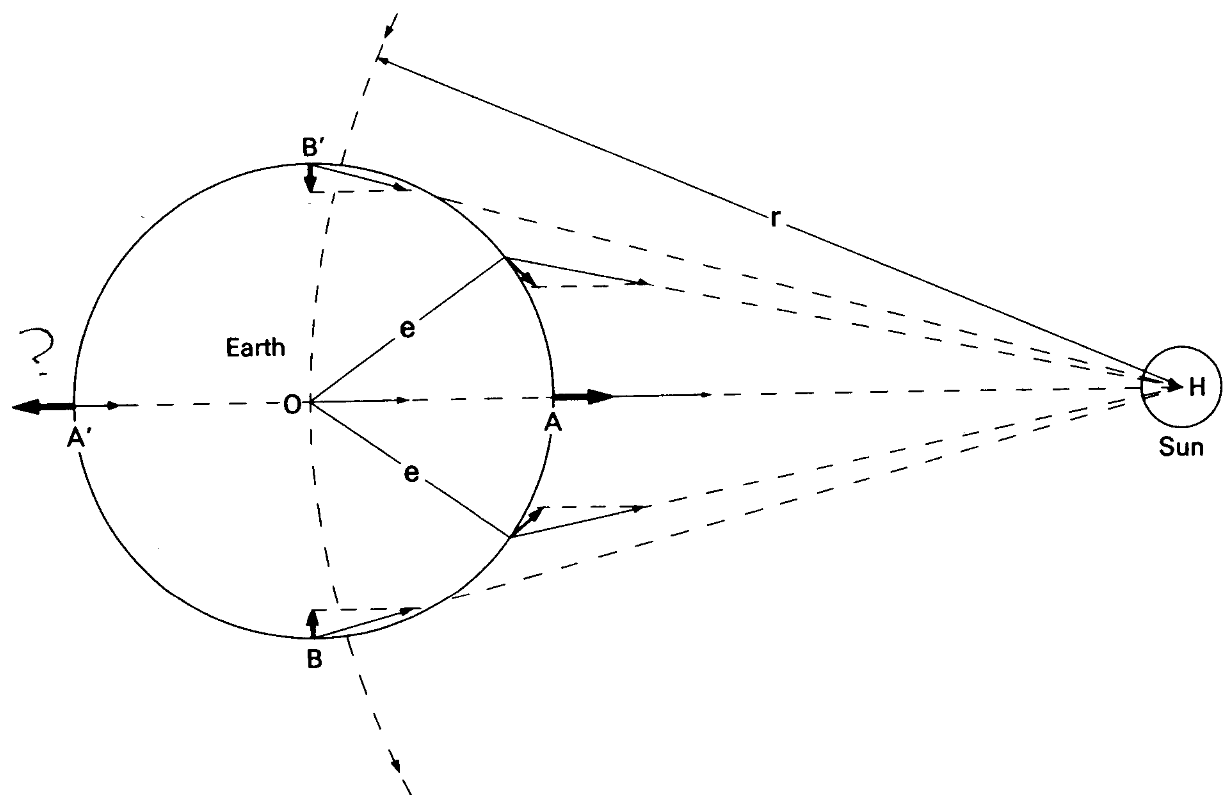

FIG. 13. Origin of sun's tide-raising forces (heavy arrows) as differences between sun's gravitational attraction and earth's centripetal acceleration in solar orbit.

tractive forces since it is they that accelerate water away from B and B' toward A and A' in an attempt to bring the surface everywhere normal to the vector sum of gravity and the tide-raising force. This ideal surface, referenced to the mean sea level defined by gravity alone, is called the *equilibrium tide*. To picture the sun's equilibrium tide in three dimensions, imagine the shapes traced out by revolving Fig. 13 about the axis AA'. Ocean tides are significant mostly because the water moves relative to the solid surface of the earth. If the earth were sufficiently pliable, it too would change shape to conform to the equilibrium tide surface, and there would be little or no relative movement of the water. The earth is not perfectly rigid, and does change shape slightly in response to the tidal forces, but these *earth tides* are small enough to neglect in this mostly qualitative discussion.

We will now estimate the magnitude of the tide-raising forces. As already stated, the sun's gravitational attraction at O in Fig. 13 is GS/r^2. At A it is $GS/(r-e)^2$, at A' it is $GS/(r+e)^2$, and at B and B' it is $GS/(r^2+e^2)$, where e is the earth's radius. All the attractions are directed from the point toward the sun's centre H. Since the tide-raising force at a point is the difference between the sun's local attraction and its attraction at the centre of the earth, we have the sun's tide-raising force, F_t, at A as

$$(2.2.3)\quad F_t(A) = \frac{GS}{r^2}\left(1 + 2\frac{e}{r} + \ldots - 1\right) \doteq 2\frac{GSe}{r^3}$$

In 2.2.3 and 2.2.4 we use the binomial expansion for $(1 - e/r)^{-2}$ and $(1 + e/r)^{-2}$ and neglect squares and higher powers of e/r, since it is so small. At A',

$$(2.2.4)\quad -F_t(A') = \frac{GS}{r^2}\left(1 - 2\frac{e}{r} + \ldots - 1\right) \doteq -2\frac{GSe}{r^3}$$

In 2.2.3 and 4, and in what follows, we have adopted the sign convention that a force directed vertically upward is positive. This explains the minus signs on the left side of 2.2.4 and 2.2.5. At B and B', the vector subtraction of the sun's attraction at O from that at B and B' gives, within the same approximation as above, only a component of the tide-raising force directed toward O, and

$$(2.2.5)\quad -F_t(B) = -F_t(B') = \left[\frac{GS}{r^2}\left(1 + \frac{e^2}{r^2}\right)^{-1}\right]\sin\beta \doteq \frac{GSe}{r^3}$$

where $\beta = \angle\ OHB = \angle\ OHB'$. In 2.2.5 we neglected e^2/r^2 and approximated $\sin\beta$ as e/r. From the above expressions we see that the tidal forces are proportional to the mass of the sun and to the inverse cube of its distance, and that the compressional forces around the great circle BB', midway between A and A', are one half the strength of the expansional forces at the points A and A', at which the sun is in the zenith and the nadir, respectively.

2.3 Moon's tide-raising force

In the previous section we spoke of the earth as orbiting around the sun, but actually the earth and the sun are both orbiting around a common centre of mass, which is less than 500 km from the centre of the sun. Similarly, the moon and the earth are orbiting about a common centre of mass, which is inside the earth, about 1 700 km beneath the surface. It is the revolution of the earth in this small orbit that is the counterpart of its revolution about the sun, which was considered in section 2.2. With this in mind, and with r as the moon's distance and M, the moon's mass, replacing S, we may apply the logic of section 2.2 directly to the earth–moon system (with H in Fig. 13 now being the moon's centre). This permits us to write down immediately expressions for the moon's tide-raising forces. The expansional forces at the points for which the moon is in the zenith and the nadir are

$$(2.3.1)\quad F_t(A) = F_t(A') \doteq 2\frac{GMe}{r^3}$$

and the compressional forces on the great circle around the earth's surface midway between these two points are

$$(2.3.2)\quad F_t(B) = F_t(B') \doteq -\frac{GMe}{r^3}$$

We have already noted in section 2.2 that the tide-raising forces of the sun are only about a half of those of the moon. It may be of some interest to compare the moon's tide-raising force to the force of gravity at the earth's surface. Neglecting the centrifugal force due to axial rotation, the surface gravity is

$$(2.3.3)\quad g = \frac{GE}{e^2} \text{ so } G = \frac{ge^2}{E}$$

where E is the earth's mass. The maximum lunar tidal force is that expressed in 2.3.1, which, with the help of 2.3.3 may be rewritten as

(2.3.4) $F_t(A) = 2g(\frac{M}{E})(\frac{e}{r})^3$

$M/E = 1/80$ and $e/r = 1/60$, which, on substitution into 2.3.4 give the maximum lunar force as $10^{-7}g$. So the tidal force is at most one ten-millionth of the earth's surface gravity. These are small forces indeed, but they act on every particle of water throughout the depth of the ocean, accelerating them toward the sublunar (or subsolar) point on the near side of the earth and toward its antipode on the far side. The undulations thus set up in the deep ocean are in fact quite gentle, and only become prominent when their energy is compressed horizontally and vertically as they ride up into shallow and restricted coastal zones.

2.4. Tidal potential and the equilibrium tide

Many force fields can be expressed as the negative gradient of a scalar field, called the potential field. Such force fields are said to be conservative, since the work done against the force in moving from a point A to a point B depends only on the positions of the two points, and not at all upon the path followed in moving between them. This constant amount of work required to move unit mass (or unit charge, etc.) from A to B is the difference in potential between A and B. The earth's gravity field is a conservative field, whose potential is given the name *geopotential*. The difference in geopotential between points is the work done against gravity in moving a unit mass from one point to the other. Equi-geopotential surfaces are the familiar level surfaces, to which free water surfaces would conform in the absence of forces other than gravity. The lunar and solar tide-raising forces are also conservative, and can be expressed as the negative gradient of the *tidal potential*. Since the sum of one or more conservative force fields can be expressed as the negative gradient of the sum of their potentials, we may add the tidal potential to the geopotential and interpret equipotential surfaces in the combined field as "level" surfaces in the combined gravity and tidal force fields. In particular, one of these equipotential surfaces would be the surface of the *equilibrium tide*, the surface to which water would conform if it could respond quickly enough to the changing tidal forces. Because the geopotential does not vary with time and because we are interested in the time-variable tides, we need only consider the tidal potential, and interpret its variations as variations in the total potential at the mean sea level.

The tidal potentials, p_t, at the point P (Fig. 14) are given very closely by the expressions

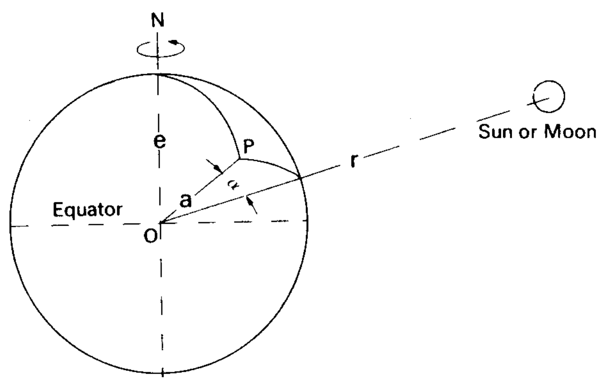

FIG. 14. Spherical triangle formed by north pole, the point on the earth directly beneath the sun (or moon) and the general point P on the earth's surface.

$$(2.4.1) \quad -p_t(\text{moon}) = \frac{GMa^2}{2r_m^3}(3\cos^2\alpha_m - 1)$$

$$-p_t(\text{sun}) = \frac{GSa^2}{2r_s^3}(3\cos^2\alpha_s - 1)$$

where r_m and r_s are the moon's and the sun's distances from the earth, the angles α_m and α_s are their zenith angles (co-altitudes), and a is the distance from the centre of the earth to the point P(equals earth's radius, e, if P is at the surface). The other symbols are as previously defined. The minus signs in front are required to conform with the convention that the force is the negative gradient of the potential. Differentiation of 2.4.1 with respect to a gives the vertical component of the tidal force. With cos $\alpha = 1$, this reproduces the expression 2.2.3 for the tidal force at A(Fig. 13), and with cos $\alpha = 0$, it reproduces the expression 2.2.5 for the tidal force at B.

The equilibrium tide surface must be an equipotential surface in the combined tidal and gravity field, and so any increase in the tidal potential must be matched by a decrease in the geopotential (i.e. a fall in the surface), and any decrease in tidal potential must be matched by an increase in geopotential (i.e. a rise in the surface). Using this fact we can calculate the height of the equilibrium tide above the mean sea level. Let the height of the equilibrium tide be Δh, which corresponds to an increase of $g\Delta h$ in geopotential. To maintain an equi-potential surface, this increase must be equal and opposite to the tidal potential, p_t, so

$$(2.4.2)\quad \Delta h = -\frac{p_t}{g}$$

Substituting the expressions for G from 2.3.3 and for p_t from 2.4.1 into 2.4.2 gives the heights of the lunar and solar equilibrium tides as

$$(2.4.3)\quad \Delta h(\text{moon}) = \frac{Me^4}{2Er_m^{\ 3}}(3\cos^2\alpha_m - 1)$$

$$\Delta h(\text{sun}) = \frac{Se^4}{2Er_s^{\ 3}}(3\cos^2\alpha_s - 1)$$

In substituting from 2.4.1 we put a equal to e because the equilibrium tide is for points at the earth's surface. The extreme values of Δh occur for $\alpha = 0°$ and $\alpha = 90°$. Using

$e = 6\,400$ km, $M/E = 0.012$,
$e/r_m = 0.017$, $S/E = 3.3 \times 10^5$,
$e/r_s = 4.3 \times 10^{-5}$,

2.4.3 gives the extreme equilibrium tide heights as

(2.4.4) $\Delta h(\text{moon}) = 0.38$ m, and -0.19 m
$\Delta h(\text{sun}) = 0.17$ m, and -0.08 m.

The ratio of the solar to the lunar values in 2.4.4 is 0.46, the same as the ratio of the extreme solar and lunar tide-raising forces (see section 2.2). In fact, the equilibrium tide reflects all the important characteristics of the tide-raising forces, and, being a scalar instead of a vector, is a much more convenient reference for local tidal observations and predictions.

2.5 Semidiurnal and diurnal equilibrium tides

The equilibrium lunar and solar surfaces defined by the expressions in 2.4.3 are ovals of revolution centred at the earth's centre and with axes directed toward the moon and the sun. They appear to rotate from east to west as the earth rotates daily on its axis with respect to the moon and the sun. The inclination of their axes north and south of the equator changes with the declination of the moon and of the sun, in a monthly cycle for the moon and an annual cycle for the sun. The ovals also change in shape as the orbital distances, r_m and r_s change in monthly and annual cycles, respectively. In formal tidal study the characteristics of the equilibrium tide are determined from mathematical analysis of expression 2.4.3 and the known astronomical parameters. It is, however, useful to obtain an intuitive understanding of how the various tidal harmonic constituents arise, and that is how we will now proceed.

Figure 15 depicts the sun's equilibrium tide superimposed on the equi-geopotential surface of mean sea level, (a) for the sun on the equator, (b) for the sun at maximum north declination, and (c) for the sun at maximum south declination. From (a) it is seen that with the sun at zero declination an observer on the equator rotates with the earth once each solar day with respect to the sun's equilibrium tide, passing through LW at points B and B' (B' is on the opposite side of the earth from B), and through HW at points A and A'. In fact, an observer at any latitude would experience equilibrium LW as his meridian passed through B and B', and HW as it passed through A and A', although the heights of

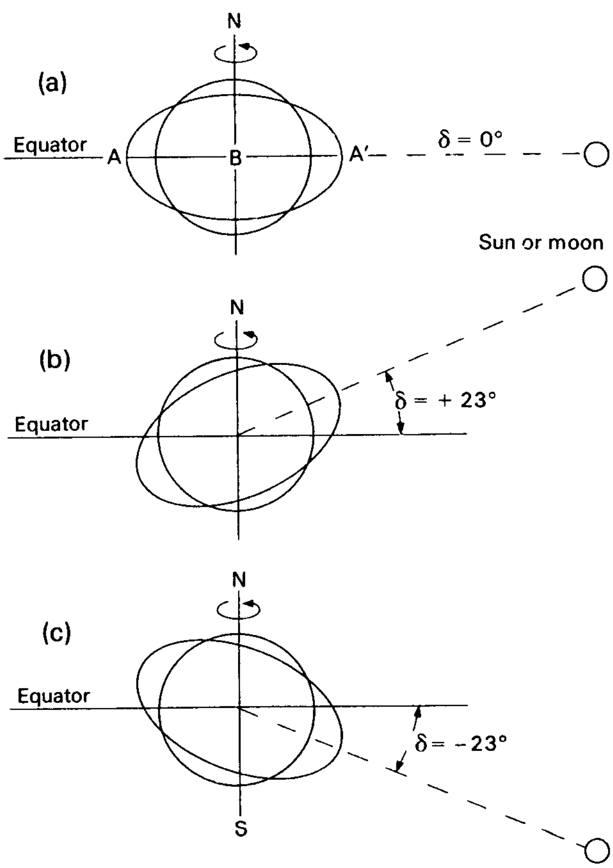

FIG. 15. Equilibrium tidal surface (AA') for sun or moon (a) on equator, (b) north of equator and (c) south of equator.

HW would decrease with increasing latitude north or south of the equator. This is the origin of the *semidiurnal solar constituent* of frequency two cycles per day (30°/h); it is designated S_2. If we simply replace the sun with the moon in the above discussion, we have the explanation for the origin of the *semidiurnal lunar constituent* (M_2). Its frequency is two cycles per lunar day (28.98°/h). The lunar day is about 50 min longer than the solar day, because the moon advances about 12.5° in its orbit each day with respect to the sun's position.

When the sun is north or south of the equator, one centre of HW for its equilibrium tide is north and the other is south of the equator, as shown in Fig. 15b and c. In these cases, an observer moving around with the earth at the equator would still experience two equal HWs and two equal LWs per day, although the HWs would not be as high as in case (a). An observer at a northern latitude would experience HHW at noon and LHW at midnight in (b) while an observer at a southern latitude would experience HHW at midnight and LHW at noon. In case (c), there would be the same inequality in the two HWs for observers away from the equator, but the northern observer would now experience HHW at midnight, etc. In the equilibrium tide the two LWs would have the same height (but not necessarily so in an actual tide). The difference in height between HHW and LHW is called the *diurnal inequality*, and it increases with the declination of the sun and with the observer's latitude (north or south) for an equilibrium tide. In fact, if the sun's declination is δ, the band of low water around the earth in the equilibrium tide extends no farther north and south than latitude 90° – δ, and observers at higher latitudes than this would see only a distorted diurnal tide, with one true HW and an extended period of low water. A semidiurnal tide with a diurnal inequality can be considered as the sum of a semidiurnal and diurnal tide. This is illustrated in Fig. 16,

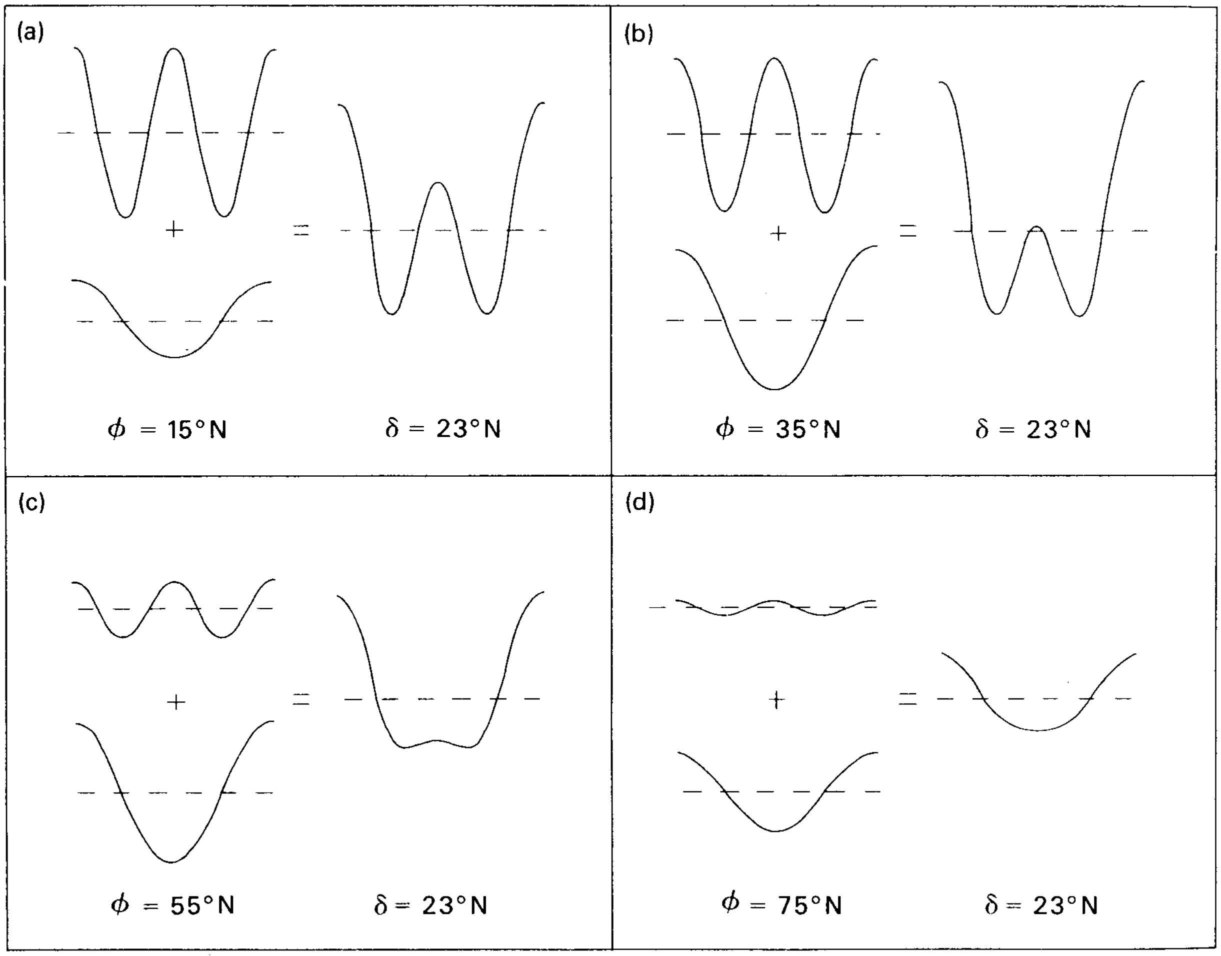

FIG. 16. Representation of solar equilibrium tide as sum of semidiurnal and diurnal contributions for sun at 23° declination, and for latitudes (a) 15°, (b) 35°, (c) 55° and (d) 75°.

which shows the combination of the semidiurnal and diurnal contributions to produce the equilibrium tide of the sun at 23° declination, for an observer at latitude (a) 15°, (b) 35°, (c) 55°, and (d) 75°.

Since the diurnal tide must reinforce the noon HW when the sun and the observer are on the same side of the equator, fall to zero when the sun is on the equator, and reinforce the midnight HW when the sun is on the opposite side of the equator, it is clear that more than a single diurnal constituent is required. From trigonometry we have the relation

$$(2.5.1)\quad \cos(n_1+n_0)t+\cos(n_1-n_0)t = 2(\cos n_0t)(\cos n_1t)$$

If n_1 is the angular speed 360° per solar day (15°/h) and n_o is 360° per year (0.04°/h), the right side of 2.5.1 is seen to be a diurnal oscillation of frequency n_1 whose amplitude is modulated at the annual frequency n_0, falling to zero at n_ot = 90° and 270°, and having maximum amplitude but with opposite phase at n_ot = 0° and 180°. Figure 17 shows a plot of 2.5.1 for a few cycles of n_1 around n_0t = 90° to

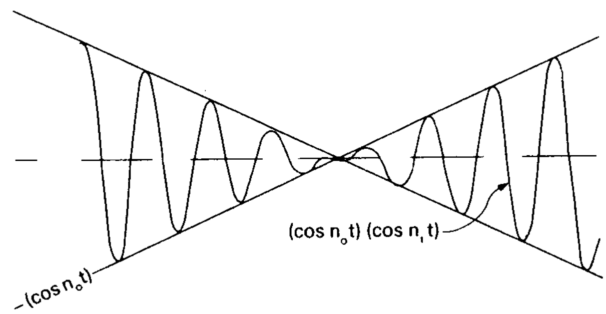

FIG. 17. Plot of $\cos(n_1+n_0)t+\cos(n_1-n_0)t = 2(\cos n_0t)(\cos n_1t)$ for a few cycles on either side of $n_0t = 90°$.

illustrate the change in amplitude and shift in phase. This is the origin of the two *solar declinational diurnal constituents*, P_1 with frequency 14.96°/h, (n_1 - n_0), and K_1 with frequency 15.04°/h, (n_1 + n_0). Because the earth's rate of rotation on its axis with respect to the "fixed stars" is equal to its rate of rotation with respect to the sun plus its rate of orbital revolution about the sun, the frequency of K_1 is seen to be one cycle per sidereal day.

The lunar equilibrium tide changes with the declination of the moon over a period of a month in the same manner as the solar tide changes with the sun's declination over a year. This, then, gives rise to two *lunar declinational diurnal constituents* with frequencies of one cycle per lunar day plus and minus one cycle per lunar month. The frequency of the earth's rotation with respect to the moon plus the moon's frequency of orbital revolution about the earth is also equal to one cycle per sidereal day, so that one of the moon's constituents has the same frequency as the corresponding one for the sun, K_1. Because of this, K_1 serves double duty, and is called the *luni-solar declinational diurnal constituent*. The other lunar constituent of the pair is O_1, with angular speed 13.94°/h.

The orbits of the moon about the earth and of the earth about the sun are not circles, but are ellipses, with the earth and the sun occupying one of the foci in the respective orbits. Thus, the distances of the moon and the sun from the earth change within the period of the particular orbit, 1 month for the moon and 1 year for the sun. The orbital points at which the moon is closest to and farthest from the earth are called *perigee* and *apogee*, respectively. The corresponding points in the earth's orbit about the sun are called *perihelion* and *aphelion*. The dependence of the tidal potential on the inverse cube of these distances (r_m and r_s in 2.4.1) causes the shape of the solar equilibrium tide (see Fig. 15) to lengthen along its axis and compress in the middle at perihelion, and to shorten along its axis and expand in the middle at aphelion. The shape of the lunar equilibrium tide changes similarly at perigee and apogee, respectively, but the change is much more pronounced for the lunar than for the solar tide. The effect of this is to modulate the amplitudes of the solar constituents with a period of a year and of the lunar constituents with a period of a month. But there is a further complication; the earth and the moon do not travel at constant angular velocities around their orbits, but travel faster when they are closer to the central body. The effect of this is to modulate the phases of the lunar constituents over a period of a month, and those of the solar constituents over a period of a year. The combined effect of the amplitude and phase modulations can be imitated mathematically by adding to each constituent two satellite constituents with frequencies equal to that of the main constituent plus and minus the orbital frequency, but with the amplitude of one satellite greater than that of the other. Figure 18 attempts to illustrate how the amplitude and phase modulations are accomplished. Tidal constituents may be regarded as rotating vectors, since they have a fixed amplitude and a uniformly increasing phase angle. A vector sum is obtained graphically by placing all of the participating vectors head to tail and joining the tail of the first to the head of the last. *OR* is the main constituent, with angular speed *n*, *RS* is the first satellite constituent, with speed $n_1 = n + n'$, and *ST* is the second satellite constituent, with speed $n_2 = n - n'$. The sum of the three constituents is *OT*, and relative to the rotating vector *OR*, the point *T* traces out an ellipse with centre at *R*. Its semi-major axis equals the sum of the satellite amplitudes, and its semi-minor axis equals their difference. It may be seen from Fig. 18

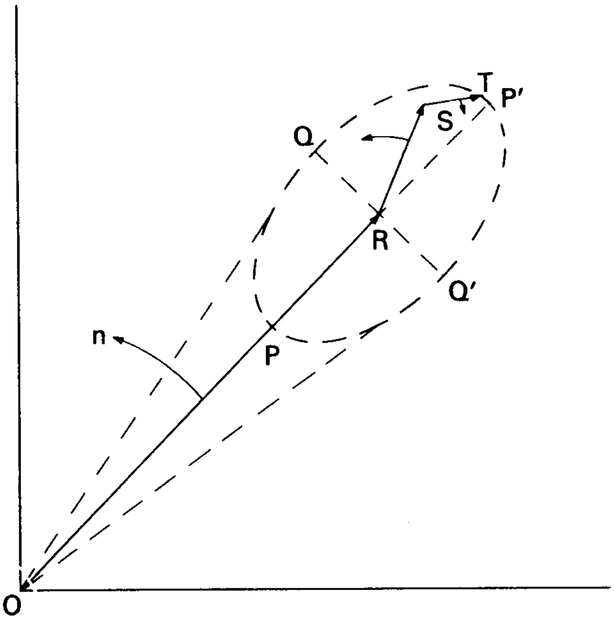

FIG. 18. Amplitude and phase modulation produced by vector sum of main constituent plus two counter-rotating satellite constituents.

that as *T* moves around the ellipse once for each cycle of the main constituent, the amplitude of the vector sum, *OT*, oscillates between *OP* and *OP'*, and the phase oscillates about that of *OR* through the angle *QOQ'*. If the satellite amplitudes are equal, the ellipse collapses to a line, and there is amplitude modulation only. This is the origin of the *larger and smaller lunar elliptic semidiurnal constituents* N_2 (speed 28.44°/h) and L_2 (speed 29.53°/h), respectively, and also of the *larger and smaller solar elliptic semidiurnal constituents* T_2 (speed 29.96°)/h) and R_2 (speed 30.04°/h). R_2 is so small it is usually neglected.

2.6 Long-period equilibrium tides

Here we will discuss tidal constituents whose periods are comparable to the sun's or the moon's orbital period. It is important to distinguish between a long-period constituent and a long-period modulation of a short-period constituent. The long-period modulation changes the range of the tide over the long period, but does not change the mean water level, whereas the long-period constituent does not change the range of the tide, but introduces a long-period fluctuation in the mean water level. To demonstrate the origin of the long-period tidal constituents we look again at Fig. 15. It is apparent that an observer near the North or South Pole will experience a lower daily average equilibrium tidal elevation when the tide-raising body (sun or moon) is on the equator as in (a) than when it is north or south of the equator as in (b) or (c). Although most easily envisaged for high latitudes, this effect is present at other latitudes as well. It results in the introduction of the lunar fortnightly constituent M_f (speed 1.10°/h), into the lunar equilibrium tide, and the solar semi-annual constituent S_{sa} (speed 0.08°/h) into the solar equilibrium tide. M_f and S_{sa} are thus related to the moon's and the sun's cyclic changes in declination. There is also a lunar monthly constituent, M_m (speed 0.54°/h), and a solar annual constituent, S_a(speed 0.04°/h), and these are related to changes in the lunar and solar distance over a month and a year, respectively.

2.7 Mathematical analysis of the equilibrium tide

In the preceding discussions we have considered the equilibrium tide as the envelope of equal tidal potential surrounding the earth at a given time. We will now express it as a time-varying function at a fixed location on the earth. To do this, we must express cos α in 2.4.3 in terms of the local latitude and of the declination and hour angle of the sun and moon. The hour angle of a celestial object is its longitude angle west of the observer's longitude. In Fig. 19, *PSN* is a spherical triangle on a sphere surrounding the earth, and with its centre at the earth's centre, *0*. *P* is the projection of the observer's position from the centre of the earth onto the sphere, *N* is the projection of the North Pole onto the sphere, and *S* is the intersection with the sphere of the line joining the centre of the earth to that of the celestial object (sun or moon). Angle *PON* is the co-latitude of *P* (90° – φ), angle SON is the co-declination of the celestial object (90° – δ), POS is its zenith angle (α) with respect to *P*, and *PNS* is

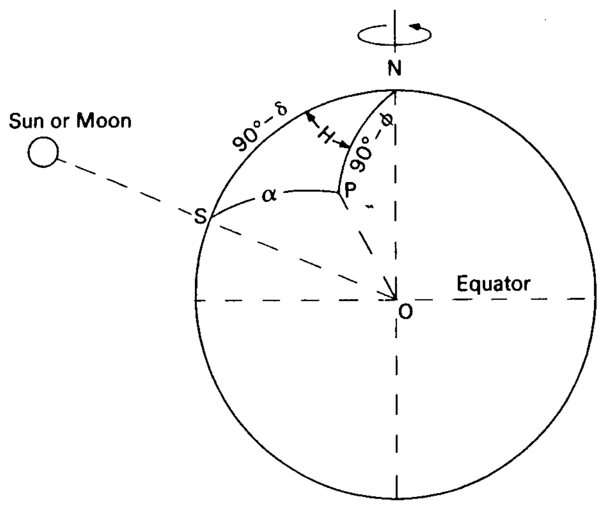

FIG. 19. Projection of the sun (or moon), the earth's polar axis, and a general point on the earth's surface onto a sphere surrounding the earth (the celestial sphere), to form the spherical triangle SNP.

its hour angle (H) with respect to P. From a formula of spherical trigonometry we have

(2.7.1) $\cos \alpha = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos H$

Substituting 2.7.1 into 2.4.3, and using some trigonometric relations to simplify it, gives the equilibrium tide at P as

(2.7.2) $$\Delta h = BC_0(\cos 2\delta - 1/3) + BC_1(\sin 2\delta) \cos H + BC_2 (\cos 2\delta + 1) \cos 2H,$$

where $B = \frac{3Me^4}{2Er_m^3}$ for the moon, $\frac{3Se^4}{2Er_s^3}$ for the sun,

and $$C_0 = 3/8 (\cos 2\varphi - 1/3)$$ $$C_1 = 1/2 \sin 2\varphi$$ $$C_2 = 1/8 (\cos 2\varphi + 1).$$

H and δ are the local hour angle and declination of the moon or the sun, as appropriate. The C_i are constants for a given latitude.

In discussing 2.7.2 we will refer only to the moon, but the same logic applies for the sun and its equilibrium tide. The first term on the right of 2.7.2 introduces the long-period species of constituent, since B varies over a period of a month and $\cos 2\delta$ varies over a period of half a month. The second term introduces the diurnal species of constituent since the hour angle (H) advances at a frequency of one cycle per lunar day. Multiplication by $\sin 2\delta$ splits the species into constituents whose frequencies differ by two cycles per month, as shown in 2.5.1 and Fig. 17 from a different approach. The third term on the right of 2.7.2 introduces the semidiurnal species of constituent, since $2H$ advances at a frequency of two cycles per lunar day. Multiplication by B modulates the species at a frequency of one cycle per month, giving rise to constituents N_2 and L_2 as defined in section 2.5. The factor $\cos 2\delta$ also modulates a portion of the semidiurnal species at a frequency of two cycles per month, introducing a pair of lunar declinational semidiurnal constituents not previously discussed. Their frequencies are two cycles per lunar day plus and minus two cycles per month. The constituent with the higher frequency is also the larger of the pair, and has the same frequency as the corresponding solar constituent, both being equivalent to two cycles per sidereal day. It is thus called the *luni-solar declinational semidiurnal constituent* or K_2 (speed 30.08°/h). Many other constituents could be discovered by examining the modulation of the declinational constituents by B and by treating the departure of some of the factors from true sinusoidal functions of time. The relative amplitudes of the constituents in the equilibrium tide can also be determined from analysis of 2.7.2 and substitution of the parameter values. The purpose of this section, however, is simply to demonstrate some of the techniques employed in identifying the important tidal frequencies. Numerical analysis on fast electronic computers now provides tools for investigation of the equilibrium tide that were not available during the early development of tidal theory. It is now quite feasible to generate from 2.7.2 the combined equilibrium tide for the sun and the moon as a time series of elevations covering many years, and to analyse it numerically into its constituents, identifying their frequency, phase, and amplitude. In Appendix A are listed a few of the equilibrium tidal constituents, along with their frequencies (as angular speed) and their amplitudes relative to that of M_2.

2.8 Spring and neap tides

It cannot be stressed too much that at no place on the earth is the actual tide the same as the equilibrium tide at that place. Nevertheless, many of the characteristics of the two are similar except for magnitude and timing. The equilibrium *spring tide* occurs on the day that the sun's and the moon's HWs fall on the same meridian, which, as shown in Fig. 20, occurs at new and full moon. The HWs occur near local noon and midnight and are higher

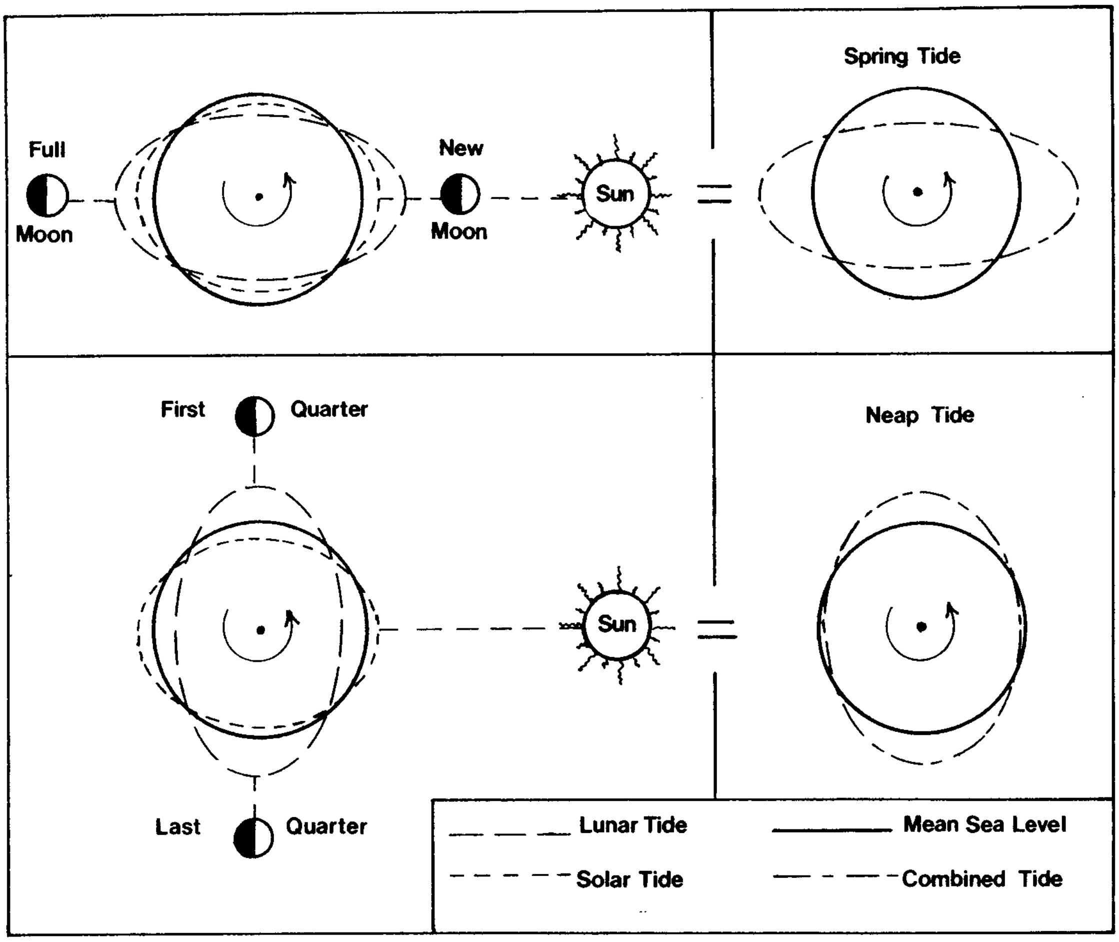

FIG. 20. Combination of lunar and solar equilibrium tides to produce spring tides at new and full moon and neap tides at moon's first and last quarter.

than average because of the reinforcement of the two. The two LWs also reinforce, but in the opposite sense, making them lower than average. The result, then, is a larger than average range of equilibrium semidiurnal tide at spring tide. The equilibrium *neap tide* occurs on the day that the sun's and the moon's HWs most closely coincide with the other's LWs, which, as shown in Fig. 20, occurs at the moon's first and last quarter. The result is a smaller than average range of tide at neap tide. The range of the combined solar and lunar tide does not, of course, change suddenly at spring and neap, but is modulated sinusoidally over the half-month period between successive springs or neaps. From the standpoint of tidal constituents, it is the interaction of M_2 and S_2 as they come in and out phase with each other that produces the springs and neaps. This fortnightly modulation of the semidiurnal tide is prominent in actual tide records as well as in the equilibrium tide, so much so in fact that in many parts of the world HW and LW at springs are taken as standards of extreme high and low waters. This is not invariably the case, however, because in other parts of the world there may be another characteristic tide that dominates over the spring tide. In the Bay of Fundy, for example, the *perigean tides* (the large semidiurnal tides associated with the moon's perigee) are equally as significant as the spring tides. In regions such as Canada's Pacific coast and parts of the Gulf of St. Lawrence it is the diurnal inequality that renders the simple spring tide heights unsatisfactory as standards of extreme high and low water.

2.9 Classification of tides

Tides are frequently classified according to the diurnal inequality they display, as a means of providing a simple description of the character of the tide in various regions. The formal classification is usually made on the basis of the ratio of some combination of the diurnal harmonic constituents over a combination of the semidiurnal constituents. A criterion that is commonly used is the ratio of the amplitude sum of O_1 and K_1 over the amplitude sum of M_2 and S_2. The ratio that is used in Canadian tidal classification is more complicated than this, but the principle is the same — the larger the numerical value of the ratio, the larger the diurnal inequality. The purpose of defining a ratio is to automate the classification once the constituents are known, avoiding the need to scan long periods of record visually. Regardless of the method used, the intent is to classify tides into four groups, qualitatively described as follows:

Semidiurnal (SD): two nearly equal HWs and two nearly equal LWs approximately uniformly spaced over each lunar day.
Mixed, mainly semidiurnal (MSD): two HWs and two LWs each lunar day, but with marked inequalities in height and irregularities in spacing.
Mixed, mainly diurnal (MD): sometimes two unequal HWs and LWs at irregular spacing over a lunar day, and sometimes only one HW and one LW in a day.
Diurnal (D): only one HW and one LW each lunar day.

Since the equilibrium tide is the same for all points at the same latitude, the earth could be divided into bands of latitude conforming to the above classification, with equilibrium tides at latitudes less than 10° being *SD*, those between latitudes 10° and 40° being *MSD*, those between 40° and 60° being *MD*, and those at latitudes higher than 60° being *D*. The actual tides, of course, may reflect the character of tide waves propagated from far away, and should not be expected to conform to the same classifications within latitude bands. Figure 21 shows sample tidal curves for one month at four Canadian locations to illustrate the four classes defined above. It is noted that all four locations lie within the same three-degree band of latitude. Figure 22 indicates on a map of Canada the regions in which the various types of tide dominate. Although the East Coast is predominantly a region of mainly semidiurnal tide, we find the only examples of the diurnal tide in the Gulf of St. Lawrence. This is because these points lie near an amphidromic point of the semidiurnal tide. We also note that the tide observed in the Arctic archipelago is heavily semidiurnal in character, unlike the equilibrium tide for these latitudes. This is because much of the tide in the Canadian Arctic has propagated in through the passages from the North Atlantic Ocean.

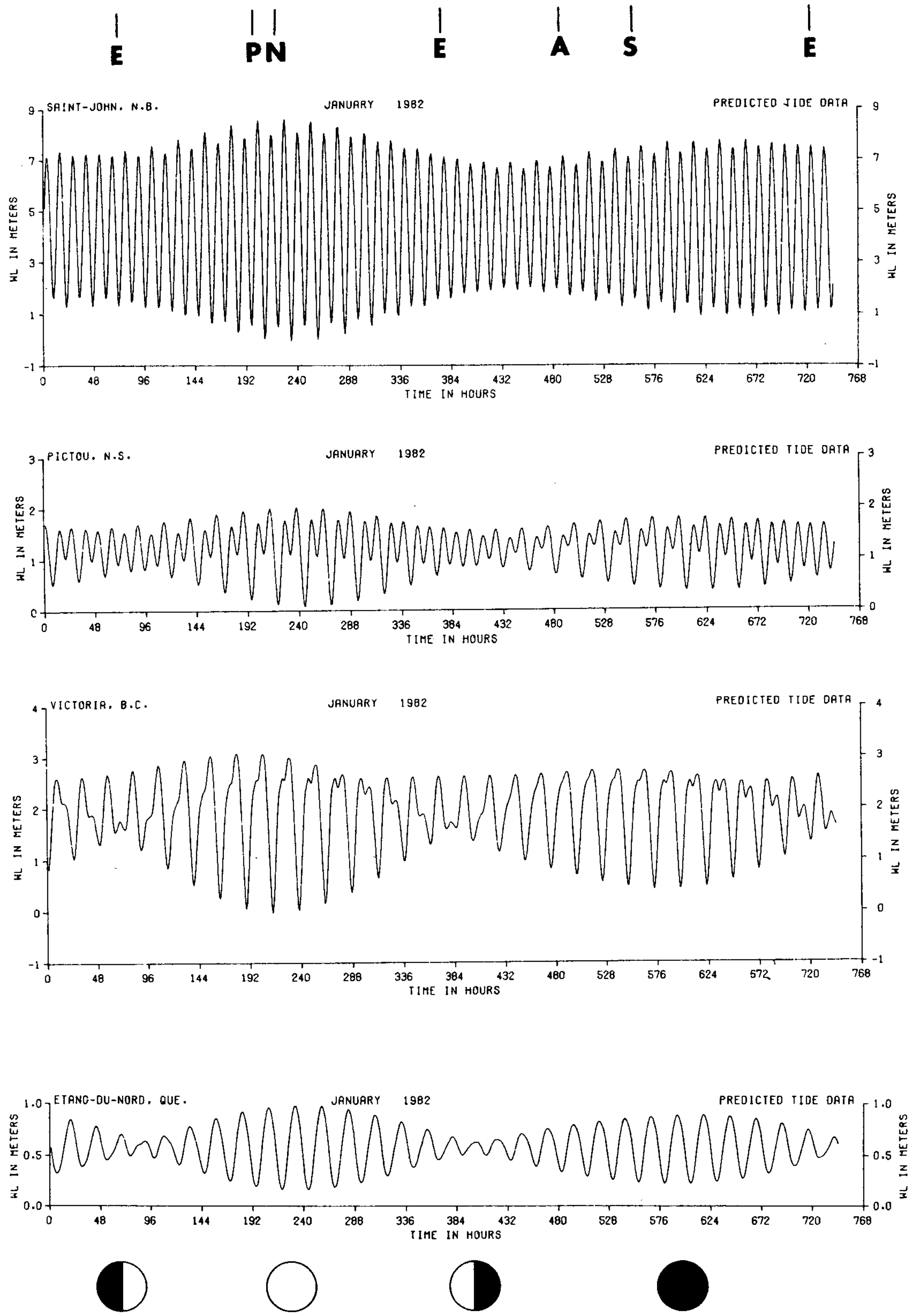

FIG. 21. Typical 1-month tidal curves from four Canadian sites, illustrating classes of tide. (a) is SD, (b) is MSD, (c) is MD and (d) is D. Letters A and P indicate apogee and perigee. E, N, and S indicate the moon is on, north, or south of equator. The circles indicate the moon's phases.

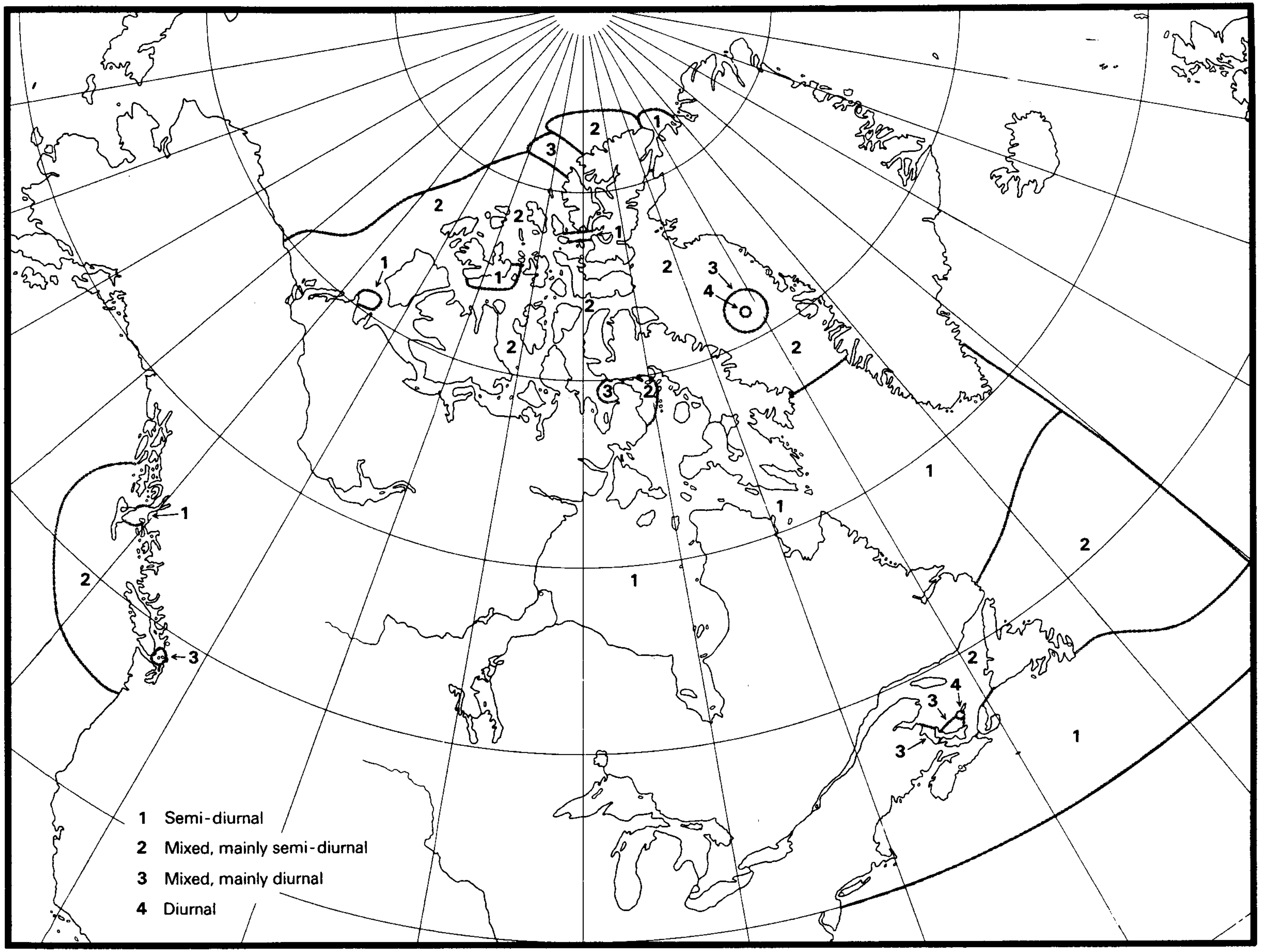

FIG. 22. Classification of tides at locations in Canadian waters.